数理解析研究所講究録 1303 巻 2003 年 56-63

COMMUTATIVITY AND NON-COMMUTATIVITY OF TOPOLOGICAL SEQUENCE ENTROPY ON CONTINUA

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ABSTRACT. Let $h_S(f)$ denote the topological sequence entropy of f respect to the sequence S. We will prove the following.

(1) $h_S(f\circ g) = h_S(g\circ f)$ for any sequence S and any graph maps f, g.
(2) For each n-dimensional compact topological manifold M with $n > 1$, there exist two continuous maps $\tilde{F}, \tilde{G} : M \to M$ such that $0 = h_{S_2}(\tilde{F}\circ\tilde{G}) < \log 2 \leq h_{S_2}(\tilde{G}\circ\tilde{F})$ and $0 = h_{S_2}(\tilde{F}|_{\Omega(\tilde{F})}) < \log 2 \leq h_{S_2}(\tilde{F})$, where $S_2 = (2^i)_{i=1}^{\infty}$ and $\Omega(\tilde{F})$ is the set of nonwandering points of $\tilde{F}$.
(3) A graph map f is chaotic in the sense of Li-Yorke if and only if the shift map $\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$ is chaotic in the sense of Li-Yorke.
(4) For any n-dimensional compact topological manifold M with $n \geq 2$, we construct a chaotic map f_M in the sense of Li-Yorke from M to itself such that the shift map σ_{f_M} is not chaotic in the sense of Li-Yorke.

(1) and (2) are the affirmative answers of questions in [BCL, Remark 4.7].

1. INTRODUCTION.

T. N. T. Goodman introduced in [G] the notion of topological sequence entropy as an extension of the concept to topological entropy. Let f be a continuous map from a compact metric space (X, d) to itself. Let $h_S(f)$ denote the topological sequence entropy of f respect to the sequence S and $h(f)$ denote the topological entropy of f. We know that if $S = (i)_{i=1}^{\infty}$, then $h_S(f)$ is equal to $h(f)$ for all continuous map f.

A map $f : X \to X$ is said to be *chaotic in the sense of Li-Yorke* if there exists an uncountable set D such that

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \text{ and } \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

for all $x, y \in D$ with $x \neq y$. This set D is called a *scramble set* of f. When X is a compact interval or the circle to itself, if $h(f) > 0$, then f is chaotic in the sense of Li-Yorke, but the converse is not true, that is, there exists a continuous map $f' : [0,1] \to [0,1]$ with $h(f') = 0$ which is chaotic in the sense of Li-Yorke. In [FS] and [H] it was proved that f is chaotic in the sense of Li-Yorke if and only if $h_S(f) > 0$ for some sequence S. This shows that chaotic maps can be characterized by the topological sequence entropy.

First, Kolyada and Snoha proved in [KS, Theorem A] that $h(f\circ g) = h(g\circ f)$ for all continuous maps f, g from a compact metric space X to itself. Moreover, it is showed in [BCL, Theorem 3.1 and Proposition 3.2] that $h_S(f\circ g) = h_S(g\circ f)$

for any sequence S if the maps f, g are onto or X is a compact interval. But, by [BCL, Theorem 4.5], there exist a 0-dimensional compact metric space X and two continuous maps $f, g : X \to X$ such that $0 = h_{S_2}(f \circ g) < h_{S_2}(g \circ f) = \log 2$, where $S_2 = (2^i)_{i=1}^{\infty}$. The first aim of this paper is to show that $h_S(f \circ g) = h_S(g \circ f)$ for any sequence S and any continuous maps f, g from a graph to itself. For any n-dimensional compact topological manifold M with $n \geq 2$, the second aim of this paper is to construct two continuous maps $\tilde{F}, \tilde{G}$ from M to itself such that $0 = h_{S_2}(\tilde{F} \circ \tilde{G}) < \log 2 \leq h_{S_2}(\tilde{G} \circ \tilde{F})$. There are the affirmative answers of questions in [BCL, Remark 4.7].

If $\Omega(f)$ denotes the set of nonwandering points of f, it is known that $\Omega(f)$ is an invariant set for f, $\Omega(f) \subset \bigcap_{n=1}^{\infty} f^n(X)$ and $h(f) = h(f|_{\Omega(f)})$, where $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ is the restriction map. Szleuk in [S] first pointed out that the formula $h_S(f) = h_S(f|_{\Omega(f)})$ does not necessarily hold. In [BCL, p.1708], it was shown that $\log 2 = h_{S_2}(f) > h_{S_2}(f|_{\Omega(f)}) = 0$ for some continuous map f from a 0-dimensional compact metric space to itself. And by [C2], there exists a continuous map $f : [0,1] \to [0,1]$ such that $h_{S_2}(f) \geq \log 2 > h_{S_2}(f|_{\Omega(f)}) = 0$. We show that for the map $\tilde{F}$ above, $h_{S_2}(\tilde{F}) \geq \log 2 > h_{S_2}(\tilde{F}|_{\Omega(\tilde{F})}) = 0$.

We define the *inverse limit space* associated to X and f to be the set

$$\varprojlim(X, f) = \{(x_i)_{i=0}^{\infty} \in X^{\infty} | f(x_i) = x_{i-1} \text{ for each } i = 1, 2, \dots\}$$

with a metric $\tilde{d}$ as $\tilde{d}((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty}) = \sum_{i=0}^{\infty} 2^{-i} d(x_i, y_i)$. And the *shift map* $\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$ is defined by

$$\sigma_f((x_i)_{i=0}^{\infty}) = (f(x_0), x_0, x_1, \dots).$$

Rongbao in [R] proved that if f is surjective, then f is chaotic in the sense of Li-Yorke if and only if σ_f is chaotic in the sense of Li-Yorke. But Cánovas in [C1] showed that the hypothesis that f is surjective can not be removed, that is, there exists a chaotic map g in the sense of Li-Yorke from 0-dimensional compact metric space to itself such that σ_g is not chaotic in the sense of Li-Yorke. And he also proved in [C1] that $f : [0,1] \to [0,1]$ (whether f is surjective or not) is chaotic in the sense of Li-Yorke if and only if σ_f is chaotic in the sense of Li-Yorke. For any n-dimensional compact topological manifold M with $n \geq 2$, from the composition method of the map $\tilde{F}$ above, we construct a chaotic map f_M in the sense of Li-Yorke from M to itself such that σ_{f_M} is not chaotic in the sense of Li-Yorke. And we show that $f : G \to G$ from a graph to itself is chaotic in the sense of Li-Yorke if and only if σ_f is chaotic in the sense of Li-Yorke.

2. Definitions.

Definition 2.1. A *continuum* is a nonempty, compact, connected, metric space. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are disjoint or intersect only in one or both of their end points.

Definition 2.2. Let Y be a subspace of a metric space X. $\mathrm{Cl}(Y)$ and $\mathrm{diam}Y$ denote the closure and the diameter of Y in a space X, respectively.

The cardinality of a set P will be denoted by $\mathrm{Card}(P)$. Let $S_k = (k^i)_{i=1}^{\infty}$ for each positive integer $k > 1$.

Let f be a continuous map from a compact metric space X to itself. We denote the n-fold composition f^n of f with itself by $f \circ \cdots \circ f$ and f^0 the identity map. Let us denote $f^{-i}(Y)$ the ith inverse image of an arbitrary set $Y \subset X$ and $f^{\omega}(X) = \bigcap_{n=1}^{\infty} f^n(X)$.

Let $\mathbf{A}, \mathbf{B}$ be finite open covers of X. Denote $\{f^{-m}(A) | A \in \mathbf{A}\}$ by $f^{-m}(\mathbf{A})$ for each positive integer m. The *mesh* of an open cover $\mathbf{A}$ of X is the supremum of the diameter of the elements of $\mathbf{A}$, denoted by $\mathrm{mesh}\mathbf{A}$. Let us define $\mathbf{A} \vee \mathbf{B} = \{A \cap B | A \in \mathbf{A}, B \in \mathbf{B}\}$ and $N(\mathbf{A})$ denotes the minimal possible cardinality of a subcover chosen from $\mathbf{A}$.

Definition 2.3. Let f be a continuous map from a compact metric space (X, d) to itself and $S = \{s_i | i = 1, 2, \dots\}$ an increasing unbounded sequence of positive integers. We define the *topological sequence entropy of f relative to a finite open cover $\mathbf{A}$ of X (respect to the sequence S)* as

$$h_S(f, \mathbf{A}) = \limsup_{n \to \infty} \frac{1}{n} \log N\Big(\bigvee_{i=1}^{n-1} f^{-s_i}(\mathbf{A})\Big).$$

And we define the *topological sequence entropy of f (respect to the sequence S)* as

$$h_S(f) = \sup\{h_S(f, \mathbf{A}) | \mathbf{A} \text{ is a finite open cover of } X\ \}.$$

If $s_i = i$ for each i, then $h_S(f)$ is equal to the standard topological entropy $h(f)$ of f introduced by Adler, Konheim and McAndrew in [AKM].

3. The graph maps case.

Lemma 3.1. *Let f be a continuous map from a graph X to itself such that $f^{\omega}(X) \neq f^n(X)$ for all n, $\mathfrak{C}_n$ the set of all components of $f^n(X) \setminus f^{\omega}(X)$ and $E_n = \bigcup\{\mathrm{Cl}(C) \cap f^{\omega}(X) | C \in \mathfrak{C}_n\}$. There exists a positive number N such that $E_n = E_N$ and $\mathrm{Card}\mathfrak{C}_n = \mathrm{Card}\mathfrak{C}_N$ for all $n \geq N$, and that $\mathrm{Cl}(C)$ is an arc and $E_N \cap \mathrm{Cl}(C)$ is one point for all $n \geq N$ and all $C \in \mathfrak{C}_n$ and that $f(E_N) = E_N$.*

By making use of Lemma 3.1, we can prove the following.

Theorem 3.2. *Let f be a continuous map from a graph X into itself. Then $h_S(f) = h_S(f|_{f^{\omega}(X)})$ for any sequence S, where $f|_{f^{\omega}(X)} : f^{\omega}(X) \to f^{\omega}(X)$ is the restriction map.*

By Theorem 3.2 and [BCL,Proposition 3.2], we have the following.

Corollary 3.3. *If f, g are continuous maps from a graph to itself, then $h_S(f \circ g) = h_S(g \circ f)$ for any sequence S.*

4. The compact set of [0,1] case.

Let us denote three Cantor sets Σ', Σ_1, and Σ_2 by $\{-2,-1,0,1,2\}^\infty, \{-1,0,1\}^\infty$, and $\{(2,\alpha_1,\alpha_2,\dots) \in \Sigma' | (\alpha_i)_{i=1}^\infty \in \Sigma_1\}$, respectively. And let $\Sigma = \Sigma_1 \cup \Sigma_2$, $\mathbf{0} = (0,0,\dots)$ and $\mathbf{1} = (1,1,\dots)$. The *shift* map $\sigma : \Sigma' \to \Sigma'$ is defined by $\sigma((\alpha_i)_{i=1}^\infty) = (\alpha_{i+1})_{i=1}^\infty$. Let $p_n : \Sigma' \to \{-2,-1,0,1,2\}^n$ be the projection for each n such that $p_n((\alpha_i)_{i=1}^\infty) = (\alpha_1,\alpha_2,\dots,\alpha_n)$ for any $(\alpha_i)_{i=1}^\infty \in \Sigma'$. Denote $\Sigma^{(n)} = p_n(\Sigma), \Sigma_i^{(n)} = p_n(\Sigma_i)$ and $0^{(n)} = (0,0,\dots,0), 1^{(n)} = (1,1,\dots,1) \in \Sigma^{(n)}$ for each $n \geq 1$ and each $i = 1,2$. For $\alpha = (\alpha_i)_{i=1}^\infty \in \Sigma$, denote $\alpha|_n = p_n(\alpha) \in \Sigma^{(n)}$ and $\Sigma_{\alpha|_n} = p_n^{-1}(\alpha|_n)$. For $\theta = (\theta_1,\theta_2,\dots,\theta_n) \in p_n(\Sigma')$ and $\theta' = (\theta'_1,\theta'_2,\dots,\theta'_{n'}) \in p_{n'}(\Sigma')$ (or $\theta' \in \Sigma'$, respectively), denote $|\theta| = (|\theta_1|,|\theta_2|,\dots,|\theta_n|)$ and $\theta * \theta' = (\theta_1,\theta_2,\dots,\theta_n,\theta'_1,\theta'_2,\dots,\theta'_{n'}) \in p_{n+n'}(\Sigma')$ (or $\theta * \theta' = (\theta_1,\theta_2,\dots,\theta_n,\theta'_1,\theta'_2,\dots) \in \Sigma'$, respectively).

Now we are going to define a *substracting machine* $\mu : \Sigma' \to \Sigma'$. First, define $\mu(\mathbf{0}) = \mathbf{1}$. Let $\alpha = (\alpha_i)_{i=1}^\infty \in \Sigma' \setminus \{\mathbf{0}\}$ and $k = \min\{i | \alpha_i \neq 0\}$. Define $\mu(\alpha) = (\mu(\alpha)_i)_{i=1}^\infty$ by

$$\mu(\alpha)_i = \begin{cases} 1 & \text{if } 1 \leq i \leq k-1 \\ 1 - |\alpha_k| & \text{if } i = k \\ \alpha_i & \text{if } i > k \end{cases}$$

We notice that

(4.1) $\mu(\Sigma_{\alpha|_n}) \subset \Sigma_{\mu(\alpha)|_n}$ for each $\alpha \in \Sigma$ and each $n \geq 1$, thus, μ is continuous.

Thus, for each $n \geq 1$, we can think of μ as a map from $\Sigma^{(n)}$ to itself defined by $\theta \mapsto \mu(\theta * \mathbf{0})|_n$. And we have

(4.2) $\mu(\theta) * \mathbf{0} = \mu(\theta * \mathbf{0})$ for all $\theta \in \Sigma^{(n)} \setminus \{0^{(n)}\}$ and

(4.3) $\mu^{2^n}(\Sigma_\theta) = \Sigma_{|\theta|}$, i.e. $\mu^{2^m}(\theta) = |\theta|$ for all $m \geq n$ and all $\theta \in \Sigma_1^{(n)}$.

Definition 4.1. (*a*) Let $\alpha, \beta \in \Sigma'$ with $\alpha|_n \neq \beta|_n$ and $k = \min\{i \leq n | \alpha_i \neq \beta_i\}$. Define $\alpha|_n < \beta|_n$ (or $\alpha < \beta$) if $\mathrm{Card}\{1 \leq i < k | \alpha_i \leq 0\}$ is even and $\alpha_k < \beta_k$ or $\mathrm{Card}\{1 \leq i < k | \alpha_i \leq 0\}$ is odd and $\alpha_k > \beta_k$.

(*b*) Let A, B be subspaces of $[0,1]$. If $x < y$ for all $x \in A$ and all $y \in B$, let us denote $A < B$.

Now we construct a family $\{D_\theta | \theta \in \Sigma^{(n)}\} (n = 1,2,\dots)$ of pairwise disjoint compact subintervals of $[0,1]$ satisfying that for any $\alpha \in \Sigma$ and any $n = 1,2,\dots$,

(4.4) $\mathrm{diam} D_{\alpha|_n} = 9^{-n}$ and

(4.5) $D_{\alpha|_{n+1}} \subset D_{\alpha|_n}$.

Moreover, we have the following property :

(4.6) $D_{\alpha|_n} < D_{\beta|_n}$ if and only if $\alpha, \beta \in \Sigma$ with $\alpha|_n < \beta|_n$.

Denote $Y_i = \bigcap_{n=1}^\infty \bigcup\{D_\theta | \theta \in \Sigma_i^{(n)}\}$ $(i = 1,2)$ and $Y = Y_1 \cup Y_2$. We see that Y_1 and Y_2 are disjoint and Cantor sets. It is known that there exists the homeomorphism $h : Y \to \Sigma$ such that $h^{-1}(\{\alpha\}) = \bigcap_{n=1}^\infty D_{\alpha|_n}$ for each $\alpha \in \Sigma$. Thus, for the sake of convenience, let us regard Y, Y_1, Y_2 and $h^{-1} \circ (\mu|_\Sigma) \circ h$ as $\Sigma, \Sigma_1, \Sigma_2$ and $\mu|_\Sigma$, respectively.

Denote $\Sigma(i) = \{\alpha \in \Sigma | \alpha_i \neq 0 \text{ and } \sigma^i(\alpha) = \mathbf{0}\}$. Let $(a_i)_{i=0}^{\infty}$ be a decreasing sequence of positive real numbers with $\sum_{i=0}^{\infty} 3^i a_i < 9^{-2}$. There exists a family $\{K_\alpha | \alpha \in \bigcup_{i=0}^{\infty} \Sigma(i)\}$ of pairwise disjoint compact subintervals of [0,1] such that $\mathrm{diam} K_\alpha < a_i$ for all $\alpha \in \Sigma(i)$ and all $i \geq 0$ and that for $\alpha, \alpha' \in \bigcup_{i=0}^{\infty} \Sigma(i)$, $\alpha < \alpha'$ implies $K_\alpha < K_{\alpha'}$.

We have a monotone map $\pi : [0,1] \to [0,1]$ with $\pi(0) = 0$ and $\pi(1) = 1$ such that

$$\pi^{-1}(x) = \begin{cases} K_\alpha & \text{if } x = \alpha \in \bigcup_{i=0}^{\infty} \Sigma(i) \\ \text{one point} & \text{if otherwise} \end{cases}$$

Denote $K_\alpha = \pi^{-1}(\alpha)$ for each $\alpha \in \Sigma$, $K_\theta = \pi^{-1}(\Sigma_\theta)$ for each $\theta \in \Sigma^{(n)}$, $\tilde{X}_1 = \pi^{-1}(\Sigma_1)$, $\tilde{X}_2 = \pi^{-1}(\Sigma_2)$ and $\tilde{X} = \pi^{-1}(\Sigma)$. By (4.6), we see that one side of $\alpha \in \Sigma$ is mapped by μ to one side of $\mu(\alpha)$. Thus, there exists the natural continuous map $\tilde{f} : \tilde{X} \to \tilde{X}$ such that $\mu \circ (\pi|_{\tilde{X}}) = (\pi|_{\tilde{X}}) \circ \tilde{f}$ and that for each $\alpha \in \bigcup_{i=1}^{\infty} \Sigma(i)$, $\tilde{f}|_{K_\alpha} : K_\alpha \to K_{\mu(\alpha)}$ is a linearly homeomorphism.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\pi|_{\tilde{X}}"'] & \tilde{X} \arrow[d, "\pi|_{\tilde{X}}"] \\ \Sigma \arrow[r, "\mu|_\Sigma"'] & \Sigma \end{tikzcd}$$

Remark 4.2. (1) We can think of $X_i = \bigcup_{\alpha \in K_i} \mathrm{Bd} K_\alpha$ $(i = 1, 2)$, $X = X_1 \cup X_2$ and $\tilde{f}$ as X_i $(i = 1, 2)$, X and $\tilde{f}$ in [BCL, p.1704], respectively.

(2) We notice that all fibers of $\pi|_X : X \to \Sigma$ have at most two points, that $(\pi|_X) \circ f = \mu \circ (\pi|_X)$ and that $h_{S_2}(\mu|_\Sigma) = 0$, but $h_{S_2}(\tilde{f}|_X) = \log 2$ by [BCL, Lemma 4.4]. This implies that Bowen's theorem (see [MS, Theorem 7.1, p.165]) for topological sequence entropy does not necessarily hold.

As the proof of [BCL, Lemma 4.3 and 4.4], we have the following.

Lemma 4.3. *With the notation above,* $0 = h_{S_2}(\tilde{f}|_{\tilde{X}_1}) < \log 2 \leq h_{S_2}(\tilde{f})$.

5. The manifolds case.

Let $a \in [0,1]^2$ and B a subspace of $[0,1]^2$. Denote $C(a, B) = \{ta + (1-t)b \in [0,1]^2 | b \in B \text{ and } t \in [0,1]\}$. If $B = \{b\}$, then we write $C(a, b) = C(a, B)$.

Let $\Sigma_1(0) = \{\mathbf{0}\} \subset \Sigma_1$, $\Sigma_1(n) = \{\alpha \in \Sigma_1 | \alpha_n \neq 0 \text{ and } \alpha_k = 0 (k > n)\} (n \geq 1)$, m_α the middle point of K_α and $b_\alpha(k) = (m_\alpha, 9^{-k}) \in [0,1]^2$ for each $\alpha \in \Sigma_1(n)$ and each $k \geq 0$. We identify $[0,1] \times \{0\}$ with $[0,1]$. Moreover let $\Lambda_\alpha = C(b_\alpha(n), K_\alpha)$ and $\Lambda_\alpha(t) = \Lambda_\alpha \cap ([0,1] \times \{t\})$ for each $\alpha \in \Sigma_1(n)$ and each $t \in [0, 9^{-n}]$.

Next, we are going to define a closed subspace $Z_1 \subset [0,1]^2$ containing $\tilde{X}_1$ and a continuous map $F_1 : Z_1 \to Z_1$ which is an extension of $\tilde{f}|_{\tilde{X}_1}$. Let $I_0 = C(b_{-1*\mathbf{0}}(0), b_{1*\mathbf{0}}(0)) \subset [0,1] \times \{1\}$. In general, for each $n \geq 1$ let

$$I_n = \bigcup_{\theta \in \Sigma_1^{(n)}} C(b_{\theta*\{-1\}*\mathbf{0}}(n), b_{\theta*\{1\}*\mathbf{0}}(n)) \subset [0,1] \times \{9^{-n}\}.$$

$$Z_1 = \tilde{X}_1 \cup \bigcup_{n \geq 0} (I_n \cup \bigcup_{\alpha \in \Sigma_1(n)} C(b_\alpha(n-1), b_\alpha(n)) \cup \Lambda_\alpha),$$

where $b_{\mathbf{0}}(-1) = b_{\mathbf{0}}(0)$. We see that Z_1 is a closed subspace and an AR by [M, Theorem 5.5.7,p.237].

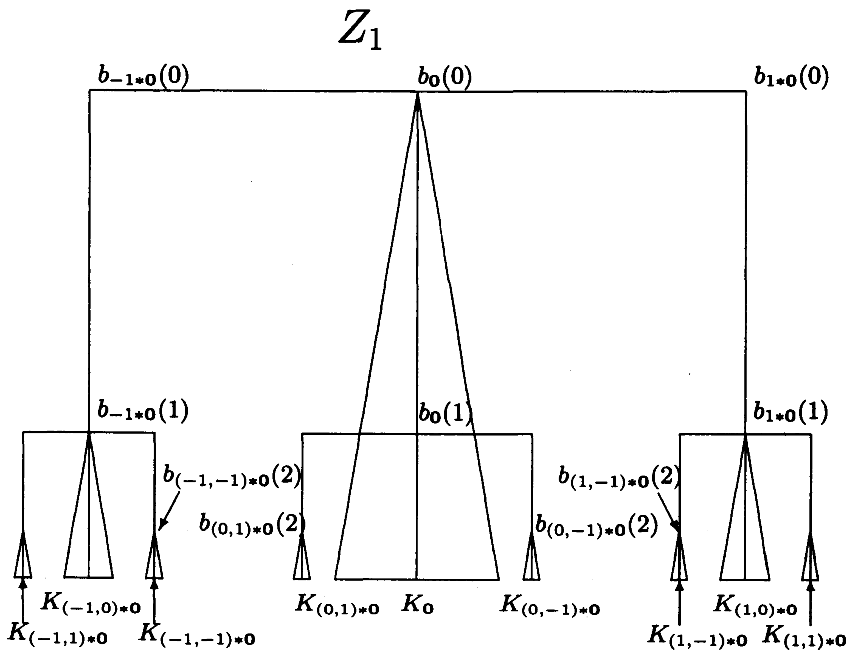

Figure.

Let us define F_1 on $\Lambda_{\mathbf{0}}$:

$F_1(\Lambda_{\mathbf{0}}(t9^{-n+1} + 5(1-t)9^{-n})) = \{tb_{1^{(n-1)}*\mathbf{0}}(n-1) + (1-t)b_{1^{(n)}*\mathbf{0}}(n-1)\}$ and
$F_1(\Lambda_{\mathbf{0}}(t5 \cdot 9^{-n} + (1-t)9^{-n})) = \{tb_{1^{(n)}*\mathbf{0}}(n-1) + (1-t)b_{1^{(n)}*\mathbf{0}}(n)\}$,

where $n \geq 1, t \in [0,1]$ and $1^{(0)} * \mathbf{0} = \mathbf{0}$. We see that $F_1(\Lambda_{\mathbf{0}})$ is the arc in Z_1 connected $b_{\mathbf{0}}(0)$ and $K_{\mathbf{1}}$.

Let us define an embedding F_1 on $C(b_\alpha(n-1), b_\alpha(n)) \cup \Lambda_\alpha$ ($\alpha \in \Sigma_1(n)$ and $n \geq 1$):

$$F_1(tb_\alpha(n-1) + (1-t)b_\alpha(n)) = tb_{\mu(\alpha)}(n-1) + (1-t)b_{\mu(\alpha)}(n) \text{ and}$$
$$F_1(tb_\alpha(n) + (1-t)x) = tb_{\mu(\alpha)}(n) + (1-t)\tilde{f}(x),$$

where $t \in [0,1]$ and $x \in K_\alpha$.

Let us define F_1 on I_0: $F_1(I_0) = \{b_{\mathbf{0}}(0)\}$.

Let us define F_1 on $C(b_{\theta*\{0\}*\mathbf{0}}(n), b_{\theta*\{\delta\}*\mathbf{0}}(n))$ ($n \geq 1, \delta = -1, 1$ and $\theta \in \Sigma_1(n)$):

$$F_1(tb_{\theta*\{0\}*\mathbf{0}}(n) + (1-t)b_{\theta*\{\delta\}*\mathbf{0}}(n)) = tb_{\mu(\theta*\{0\}*\mathbf{0})}(n) + (1-t)b_{\mu(\theta*\{\delta\}*\mathbf{0})}(n),$$

where $t \in [0,1]$.

Final, let us define F_1 on $A = C(b_{\theta*\{0\}*\mathbf{0}}(n), b_{\theta*\{\delta\}*\mathbf{0}}(n))$ $(n \geq 1, \delta = -1, 1$ and $\theta \in \Sigma_1^{(n)}$ with $\theta_n = 0)$. If $\theta_i = 0 (1 \leq i \leq n)$, define $F_1(A) = \{F_1(b_{\mathbf{0}}(n))\}$. Let $\theta_i \neq 0$ for some i. Since F_1 is defined on $(A \cap \Lambda_{\theta*\{0\}*\mathbf{0}}) \cup \{b_{\theta*\{\delta\}*\mathbf{0}}(n)\}$, we can naturally extend F_1 on A which is an embedding.

Denote $K_{\theta,m} = (K_\theta \times [0, 9^{-m}]) \cap Z_1$ for each $\theta \in \Sigma_1^{(n)}$ and each $m \geq 0$. By the definition of F_1, we have

(5.1) $F_1(Z_1 \cap [0,1] \times [9^{-m-1}, 9^m]) \subset Z_1 \cap [0,1] \times [9^{-m-1}, 9^m]$ for each $m \geq 0$ and
(5.2) $F_1(K_{\theta,m}) \subset K_{\mu(\theta),m}$ for each $\theta \in \Sigma_1^{(n)}$ and each $m \geq 1$.

Lemma 5.1. *With the notation above, $h_{S_2}(F_1) = 0$.*

Let Z_{-1} be the closure of the component of $Z_1 \setminus \{b_{\mathbf{0}}(0)\}$ containing $K_{(-1)*\mathbf{0}}$. We can construct a closed subspace $Z_2 \subset [0,1]^2$ containing $\tilde{X}_2$ and a homeomorphism $F_2 : Z_2 \to Z_{-1}$ which is an extension of $\tilde{f}|_{\tilde{X}_2}$ such that $Z_2 \cap Z_1 = \{b_{1*\mathbf{0}}(0)\}$. Define $Z = Z_1 \cup Z_2$, $F = F_1 \cup F_2 : Z \to Z$ and $G : Z \to Z$ by $G|_{Z_1} = F_1$ and $G(Z_2) = \{b_{\mathbf{0}}(0)\}$. As the proof of [BCL, Theorem 4.5], we obtain the following.

Theorem 5.2. *With the notation above, $h_{S_2}(F) \geq \log 2$ and $0 = h_{S_2}(F \circ G) < \log 2 \leq h_{S_2}(G \circ F)$.*

Since Z is an AR, by Theorem 5.2, we can prove the following.

Theorem 5.3. *For each n-dimensional compact topological manifold M with $n > 1$, there exist two continuous maps $\tilde{F}, \tilde{G} : M \to M$ such that $0 = h_{S_2}(\tilde{F} \circ \tilde{G}) < \log 2 \leq h_{S_2}(\tilde{G} \circ \tilde{F})$ and $0 = h_{S_2}(\tilde{F}|_{\Omega(\tilde{F})}) < \log 2 \leq h_{S_2}(\tilde{F})$.*

6. Some applications to inverse limit spaces

By making use of Lemma 3.1, we can prove the following.

Lemma 6.1. *Let X be a graph and f a continuous map from X to itself. Then f is chaotic in the sense of Li-Yorke if and only if $f|_{f^\omega(X)} : f^\omega(X) \to f^\omega(X)$ is chaotic in the sense of Li-Yorke.*

As in proof of [C1, Theorem 2.2], by Lemma 6.1 we can show the following.

Theorem 6.2. *Let X be a graph and f a continuous map from X to itself. Then f is chaotic in the sense of Li-Yorke if and only if σ_f is chaotic in the sense of Li-Yorke.*

Remark 6.3. Let f be a continuous map from a compact metric space X to itself. The proof of [C1, Theorem 2.2] implies that if σ_f is chaotic in the sense of Li-Yorke, then $f|_{f^\omega(X)}$ is chaotic in the sense of Li-Yorke, thus, f is chaotic in the sense of Li-Yorke.

Theorem 6.4. *For each n-dimensional compact topological manifold M with $n > 1$, there exists a continuous maps $f_M : M \to M$ such that f_M is chaotic in the sense of Li-Yorke and that σ_{f_M} is not chaotic in the sense of Li-Yorke.*

Thorem 6.4 shows the possibility of the existence of a map which is not chaotic in the sense of Li-Yorke with positive topological entropy. But, recently, F. Blanchard, E. Glasner, S. Kolyada, and A. Maass [BGKM] prove that every continous map with positive topological entropy is chaotic in the sense of Li-Yorke.

REFERENCES

[AKM] R.L. Adler, A.G. Knnhein and M.H. McAndrew, *Topological entropy,* Trans. Amer. Math. Soc., **114** (1965), 309–319.

[B] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces,* Trans. Amer. Math. Soc. **153** (1971), 401–414 ; Erratum ; Trans. Amer. Math. Soc. **181** (1973), 509–510.

[BC] L. Block and W. Coppel, *Dynamics in One Dimension,* Lecture Notes in Math. 1513, Springer-Verlag, 1992.

[BCL] F. Balibrea, J. Cánovas and V. J. López, *Commutativity and non-commutativity of topological sequence entropy,* Ann. Inst. Fourier, Grenoble, **49** (1999), 1693–1709.

[BGKM] F. Blanchard, E. Glasner, S. Kolyada, and A. Maass, *On Li-Yorke pairs,* J. reine. angew. Math., **547** (2002), 51–68.

[C1] J. Cánovas, *On topological sequence entropy and chaotic maps on inverse limit spaces,* Acta. Math. Uni. Comenianae, **68** (1999), 205–211.

[C2] J. Cánovas, *An interval counterexample on topological sequence entropy,* Acta Math. Hungar., **88** (2000), 123–131.

[C] N. Chinen, *Topological sequence entropy of monotone maps on one-dimensional continua,* to submitted.

[FS] N. Franzová and J. Smítal, *Positive sequence topological entropy characterizes chaotic maps,* Proc. Amer. Math. Soc., **112** (1991), 1083–1086.

[G] T. N. T. Goodman, *Topological sequence entropy,* Proc. London Math. Soc., **29** (1974), 331–350.

[H] R. Hric, *Topological sequence entropy for maps of the circle,* Comment. Math. Univ. Carolinae, **41** (2000), 53–59.

[KS] S. Kolyada and L'. Snoha, *Topological entropy of nonautonomous dynamical systems,* Random and Comp. Dynamics, **4** (1996), 205–233.

[LY] T. Y. Li and J. A. Yorke, *Period three implies chaos,* Amer.Math.Monthly 82 (1975), 985–992.

[M] J. van Mill, *Infinite-dimensional topology,* North-Holland Math. Library, 43., 1989.

[MS] W. de Melo and S. van Strien, *One-dimensional dynamics,* Series of Modern Surveys in Math., Springer, Berlin, 1993.

[N] S.B. Nadler Jr, *Continuum Theory An Introduction,* Pure and Appl.Math.158(1992).

[R] Gu. Rongbao, *Topological entropy and chaos of shift maps on the inverse limits spaces,* J. Wuhan Univ. (Natural Science Edition), **41** (1995), 22–26.

[S] W. Szlenk, *On weakly* conditionally compact dynamical systems,* Studia Math., **66** (1979), 25–32.

[W] P. Walters, *An introduction to ergodic theory,* Springer-Verlag, Berlin, 1982.

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